Lakshya Forum for Competitions™

MT EDUCARE

PRMO 2018

(QUESTION WITH SOLUTION)

MATHEMATICS

1. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number. What is the largest prime factor of n?

Ans. (17)

Let x, y, z be no. of pages in volumes I, II, III

$\therefore \quad y = x + 50 \quad \ldots\ldots. (1)$

and $\quad z = 1.5\,y \quad \ldots\ldots. (2)$

Also $\quad 1 + (x+1) + (x+y+1) = 1709$

$2x + y = 1706 \quad \ldots\ldots. (3)$

Solving (1) and (3)

$3x = 1706 - 50 = 1656$

$x = 552$

$y = 602$

$z = 1.5 \times 602 = 903$

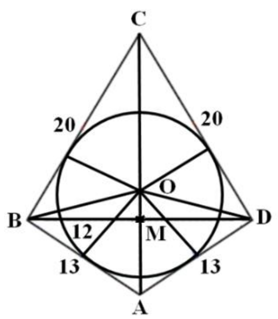
$n = x + y + z = 2057$

$= 11 \times 187$

$= 11 \times 11 \times 17$

Largest prime factor = 17.

2. In a quadrilateral ABCD. It is given that AB = AD = 13, BC = CD = 20, BD = 24. If r is the radius of the circle inscribable in the quadrilateral. then what is the integer closest to r?



Ans. (8)

$BM = \frac{1}{2} BD = 12$

$AM = 5,\ CM = 16$

Area of quadrilateral

ABCD = Area ΔABD + Area ΔBCD

$= \frac{1}{2} \times 24 \times 5 + \frac{1}{2} \times 24 \times 16$

$= 12[5 + 16] = 12 \times 21 = 252$

Also area of quad ABCD = Area ΔAOB + Area ΔBOC + Area ΔCOD + Area ΔAOD

$= \frac{1}{2} \times 13 \times r + \frac{1}{2} \times 20 \times r + \frac{1}{2} \times 20 \times r + \frac{1}{2} \times 13 \times r$

$= 33r = 252$

$r = 7.63 \approx 8$

3. Consider all 6-digit numbers of the form abccba where b is odd. Determine the number of all such 6-digit number that are divisible by 7.

Ans. (70)

$abccba = 100001a + 10010b + 1100c$

$\equiv 6a + c \pmod 7$

$(a, c) = \{(1,1), (1,8), (2,2), (2,9), (3,3), (4,4), (5,5), (6,6),$

$(7,7), (7,0), (8,8), (8,1), (9,9)(9,2)\}$

Also, b can be 1, 3, 5, 7, 9

So, Total number of no. of given form $= 14 \times 5 = 70$

4. The equation 166 × 56 = 8590 is valid in some base $b \geq 10$ (that is 1, 6, 5, 8, 9, 0 are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation.

Ans. (12)

In base 'b' to decimal system

$166 = b^2 + 6b + 6, \qquad 56 = 5b + 6$

$8590 = 8b^3 + 5b^2 + 9b$

$\therefore (b^2 + 6b + 6)(5b + 6) = 8b^2 + 5b^2 + 9b$

Solving $3b^3 - 31b^2 - 57b - 36 = 0$

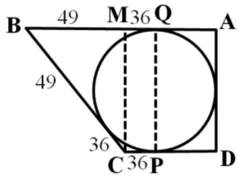
By putting b = 12 satisfies it

$3b^3 - 31b^2 - 57b - 36 = (b - 12)[3b^2 + 5b + 3] = 0$

Only possible value = 12.

5. Let ABCD be a trapezium in which AB || CD and AD ⊥ AB. Suppose ABCD has an incircle which touches AB at Q and CD at P. Given that PC = 36 and QB = 49, find PQ.

Ans. (84)



$MB = 49 - 36 = 13$ and $BC = 49 + 36 = 85$

$\therefore CM = \sqrt{85^2 - 13^2}$

$= \sqrt{98 \times 72} = 84$

$PQ = CM = 84$

6. Integers a, b, c, satisfy a + b – c = 1 and $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$?

Ans. (18)

$c = (a + b - 1)$

$a^2 + b^2 - (a + b - 1)^2 = 1$

$\rightarrow -2ab + 2a + 2b = 0$

$ab - a - b + 1 = 1$

$(a - 1)(b - 1) = 1$

So, $a - 1 = 1,\ b - 1 = 1$

$a = 2,\ b = 2,\ \rightarrow c = 3$

Or $(a - 1) = -1,\ b - 1 = -1$

$a = 0,\ b = 0\ \rightarrow\ c = -1$

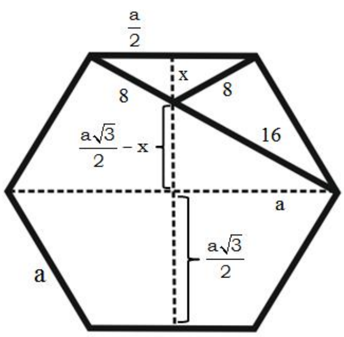
So, $a^2 + b^2 + c^2 = 17$ or 1

$Sum = 17 + 1 = 18$.

7. A point P in the interior of a regular hexagon is at distance 8,8,16 units from three consecutive vertices of the hexagon. respectively. If r is radius of the circumscribed circle of the hexagon. What is the integer closest to r?

Ans. (14)

$\left(\frac{a}{2}\right)^2 + x^2 = 64$ .....(1)



And $\left(\frac{a\sqrt{3}}{2} - x\right)^2 + a^2 = 256$ .....(2)

Multiply eq. (1) by 4 and subtract eq. (2) from (1),

$$4x^2 - \left(\frac{a\sqrt{3}}{2} - x\right)^2 = 0$$

$$2x = \frac{a\sqrt{3}}{2} - x$$

$$x = \frac{a\sqrt{3}}{6}$$

From (1) $\left(\frac{a}{2}\right)^2 + \frac{a^2}{12} = 64 \quad \rightarrow \quad \frac{3a^2 + a^2}{12} = 64$

$$a^2 = 3 \times 64$$

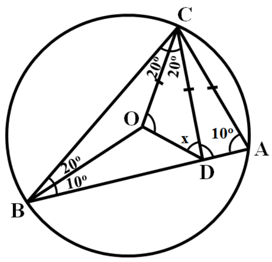
Circum-Radius = $8\sqrt{3}$ □ $13.85 \approx 14$ [Circum radius of regular hexagon = it's side]

8. Let AB be a chord of a circle centre O. Let C be a point on the circle such that ∠ABC = 30^0 and O lies inside the triangle ABC. Let D be a point on AB such that ∠DCO = ∠OCB = 20^0. Find the measure of ∠CDO in degree.

Ans. (80)

In ΔAOC

$\angle AOC = 2\angle ABC = 60°$



$\rightarrow \angle OAC = \angle OCA = 60°$

$\therefore OA = OC = CA$

In ΔACD

$\angle D = \angle DBC + \angle DCB = 30° + 40° = 70°$

$\angle A = 180° - (70° + 40°) = 70°$

$\therefore AC = CD = OA = OC$

In ΔODC, $\angle O = \angle D = x(\text{Say})$

So, $x + x + 20 = 180$

So, $x = \frac{160}{2} = 80°$

9. Suppose a, b are integer and a + b is a root of $x^2 + ax + b = 0$. What is the maximum possible value of b^2?

Ans. (81)

$(a+b)^2 + a(a+b) + b = 0$

$\rightarrow 2a^2 + 3ab + b^2 + b = 0$

$D = 9b^2 - 8(b^2 + b)$

$= b^2 - 8b = (b-4)^2 - 16 = k^2$ (perfect square)

$\rightarrow (b-4)^2 - k^2 = 16$

$\left.\begin{array}{l}\therefore (b-4)+k = a \\ (b-4)-k = b\end{array}\right\}$ where $a \cdot b = 16$

$2(b-4) = 10, -10, 8, -8$ as, $a = 8, -8, 4, -4$

$\therefore b = 9, -1, 8, 0$ $b = 2, -2, 4, -4$

For b = 9,

$2a^2 + 27a + 90 = 0$

$(a+6)(2a+15) = 0$

$a = -6$

So, max. possible value of $b^2 = 81$

10. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$.

Ans. (24)

$BC = 2\sqrt{x^2 + y^2}$

Also, $AC^2 + AB^2 = 2(AD^2 + CD^2)$

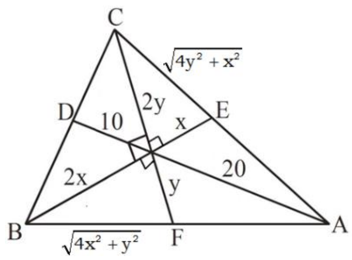
$\rightarrow 4(4y^2 + x^2) + 4(4x^2 + y^2)$

$= 2(900 + (x^2 + y^2))$

$18(x^2 + y^2) = 1800$

$x^2 + y^2 = 100$

$\frac{AB^2 + BC^2 + CA^2}{100} = \frac{24(x^2 + y^2)}{100} = 24$



11. There are several tea cups in the kitchen, some with handles and the other without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?

Ans. (29)

${}^{n}C_2 \times {}^{m}C_3 = 1200$

$\rightarrow n(n-1) \times m(m-1) \times (m-2) = 1200 \times 2! \times 3!$

$[n(n-1)][m \cdot (m-1)(m-2)] = 12 \times 12 \times 100 = 2^6 \cdot 3^2 \cdot 5^2$

R.H.S. should be product of 2 consecutive and 3 consecutive +ve integer with prime factors only 2, 3 or 5. So possibility are :

$[4 \times 5]\ [10 \times 9 \times 8] = 2^6 \cdot 3^2 \cdot 5^2$

or $[25 \times 24]\ [4 \times 3 \times 2] = 2^6 \cdot 3^2 \cdot 5^2$

$\therefore$ Max. no. of cups : 25 + 4 = 29

12. Determine the number of 8-tuples $(\epsilon_1, \epsilon_2, ... \epsilon_8)$ such that $\epsilon_1, \epsilon_2, ... \epsilon_8 \in \{1, -1\}$ and $\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + ... + 8\epsilon_8$ is a multiple of 3.

Ans. (88)

$\varepsilon_1 + 2\varepsilon_2 + 3\varepsilon_3 + ... + 8\varepsilon_8 \equiv (\varepsilon_1 + \varepsilon_4 + \varepsilon_7) + 2(\varepsilon_2 + \varepsilon_5 + \varepsilon_8) \pmod 3$

$(\varepsilon_1 + \varepsilon_4 + \varepsilon_7)$ can be +3, -3, +1, -1 (3 ways)

(in 1 way) (1 way) (3 ways) (3 ways)

$(\varepsilon_2 + \varepsilon_5 + \varepsilon_8)$ can be +3, -3, +1, -1

(ε_3 and ε_6) can be +1 or -1

So, req. ways :

$[2 \times 2 + 3 \times 3 + 3 \times 3] \times 2 \times 2 = 88$

13. In a triangle ABC, right-angled at A, the altitude through A and the internal bisector of $\angle A$ have lengths 3 and 4, respectively. Find the length of the median through A.

Ans. (24)

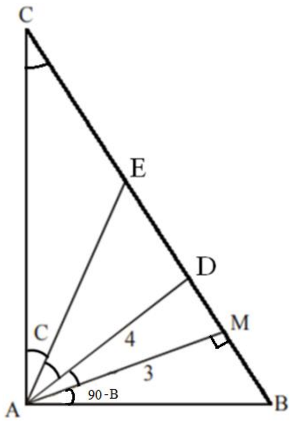
$\angle EAD = 45 - C$

$\angle DAM = B - 45$

$= (90 - C) - 45 = 45 - C$

$\therefore$ AD is also angle bisector of $\angle EAM$ $\quad DM = \sqrt{7}$

In ΔAEM, Let $AE = x$



$EM = \sqrt{x^2 - 9}$

$\frac{x}{3} = \frac{ED}{DM} = \frac{\sqrt{x^2 - 9} = \sqrt{7}}{\sqrt{7}}$

$\rightarrow \sqrt{7} \cdot x = 3\sqrt{x^2 - 9} - 3\sqrt{7}$

$\rightarrow \sqrt{7}(x + 3) = 3\sqrt{x^2 - 9}$

Squaring $7(x^2 + 9 + 6x) = 9(x^2 - 9)$

$\rightarrow 2x^2 - 42x - 144 = 0$

$x^2 - 21x - 72 = 0$

$(x - 24)(x + 3) = 0$

$x = 24$

14. If x = $\cos 1^0 \cos 2^0 \cos 3^0 ... \cos 89^0$ and y = $\cos 2^0 \cos 6^0 \cos 10^0 ... \cos 86^0$, then what is the integer nearest to $\frac{2}{7}\log_2\left(\frac{y}{x}\right)$?

Ans. (19)

AD : Angle Bisection
AM : Altitude

$$\frac{y}{x} = \frac{\cos 2^\circ \cos 6^\circ \cos 10^\circ ....\cos 86^\circ}{(\cos 1^\circ \cos 89^\circ)(\cos 2^\circ \cos 88^\circ).......(\cos 44^\circ \cos 46^\circ)\cos 45^\circ}$$

$$= \frac{(\sin 88^\circ \sin 84^\circ \sin 80^\circ ....\sin 4^\circ) \times 2^{44}}{(2\cos 1^\circ \sin 1^\circ)(2\cos 2^\circ \sin 2^\circ).......(2\cos 44^\circ \sin 44^\circ)} \times \sqrt{2}$$

$$= \frac{(\sin 88^\circ \sin 84^\circ \sin 80^\circ ....\sin 4^\circ) 2^{44} \times \sqrt{2}}{\sin 2^\circ \sin 4^\circ ....\sin 88^\circ}$$

$$= \frac{(\sin 88^\circ \sin 84^\circ \sin 80^\circ ....\sin 4^\circ)2^{44.5}}{(\sin 2^\circ \sin 88^\circ)(\sin 4^\circ \sin 86^\circ).......(\sin 44^\circ \sin 46^\circ)}$$

$$= \frac{(\sin 88^\circ \sin 84^\circ \sin 80^\circ ....\sin 4^\circ)2^{44.5} \cdot 2^{22}}{(2\sin 2^\circ \cos 2^\circ)(2\sin 4^\circ \cos 4^\circ).......(2\sin 44^\circ \cos 44^\circ)} = 2^{66.5}$$

$$\therefore \frac{2}{7}\log\frac{y}{2x} = \frac{2}{7}\times 66.5 = 19$$

15. Let a and b be natural numbers such that 2a – b, a – 2b and a + b are all distinct squares. What is the smallest possible value of b?

Ans. (21)

As (2a – b) = (a – 2b) + (a + b)

So they are Pythagorean triples

$2a - b = H^2$ .... (1)

$a - 2b = B^2$ .... (2)

$a + b = P^2$ .... (3)

$3b = P^2 - B^2$

$$b = \frac{P^2 - B^2}{3}$$

$\because$ In Pythagorean triples atleast one of P or B is multiple of ‘3’.

$\therefore$ Here P and B both are multiple of 3.

For smallest value of b

Take P = 12, B = 9

$$b = \frac{144 - 81}{3} = 21$$

Also, for this a = $123 \in N$.

16. What is the value of $\sum\limits_{\substack{1 \le i < j \le 10 \\ i+j = odd}} (i + j) - \sum\limits_{\substack{1 \le i < j \le 10 \\ i+j = even}} (i + j)$?

Ans. (55)

Method – 1

Number of ways of $[(i + j) = odd] = \frac{10 \times 5}{2} = 25$

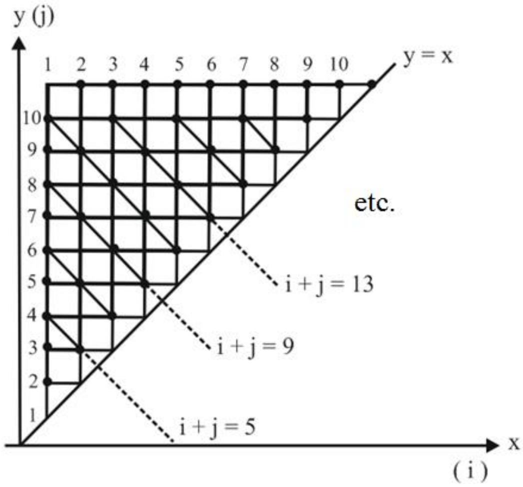
Number of ways of $[(i + j) = even] = \frac{10 \times 4}{2} = 20$ $\qquad \because (i \ne j)$

Now, for every (a, b) such that a < b and a + b = even we can have (a + 1, b) such that a + 1 < b and a + 1 + b = odd.

So for 20 pair of (a, b) [a + b = even] difference with corresponding (a + 1, b) = 20

Also other 5 ordered pair (1, 2), (1, 4), (1, 6), (1, 8), (1, 10) are not corresponding to any (a, b), (a + b = even)

Their sum = 5 + 30 = 35



$\therefore$ Req. diff = 20 + 35 = 55

Method – 2

$$\sum_{\substack{1\le i<j\le 10\\ i+j=\text{odd}}}(i+j) = 1\times3+2\times5+3\times7+4\times9+5\times11 +4\times13+3\times15+2\times17+1\times19$$

Similarly

$$\sum_{\substack{1\le i<j\le 10\\ i+j=\text{even}}}(i+j) = 1\times4+2\times6+3\times3+4\times10+4\times12 +3\times14+2\times16+1\times18$$

$$\sum_{(\text{odd})}(i+j) - \sum_{(\text{even})}(i+j) = 55$$

17. Triangles ABC and DEF are such that $\angle A = \angle D$, AB = DE = 17. BC = EF = 10 and AC – DF = 12. A What is AC + DF?

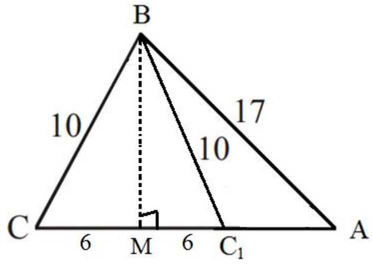
Ans. (30)

Method – 1

$BM = \sqrt{100-36} = 8$

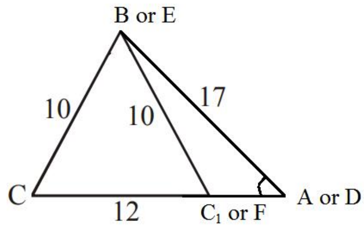
$AM = \sqrt{17^2 - 8^2} = 15$

$\therefore AC + AC_1 = (15+6)(15-6) = 30$



Method – 2

$$\cos A = \frac{289 + b^2 - 100}{2 \times 17 \times b}$$



Where

b : AC or AC_1

$\rightarrow b^2 + 189 = 34\, b \cdot \cos A$

$\rightarrow b^2 + 34\, b \cdot \cos A + 189 = 0 < \begin{matrix} b_1 \\ b_2 \end{matrix}$

$(b_1 + b_2)^2 = (b_1 - b_2)^2 + 4\, b_1 b_2$

$= 12^2 + 4 \times 189 = 900$

$\therefore b_1 + b_2 = 30$

18. If $a, b, c \geq 4$ are integers, not all equal, and $4abc = (a+3)(b+3)(c+3)$, then what is the value of $a+b+c$?

Ans. (16)

$4\, abc = (a+3)(b+3)(c+3) \qquad a \geq 4$

$\rightarrow \frac{a+3}{a} \times \frac{b+3}{b} \times \frac{c+3}{c} = 4 \qquad \text{.... (1)}$

If we take a = 4, then one of b or c must be 7.

i.e. $\frac{7}{4} \times \frac{10}{7} \times \frac{c+3}{3} = 4$

$\rightarrow c = 5$

Take higher values of a, b, c will decrease the product of L.H.S of (1)

So, a = 4, b = 7, c = 5

$\therefore a + b + c = 16$

19. Let $N = 6 + 66 + 666 + \ldots + 66\ldots66,$ where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N?

Ans. (33)

$N = \frac{6}{9}[(10-1) + (10^2 - 1) + \ldots\ldots (10^{100} - 1)]$

$= \frac{2}{3}\left[\frac{10(10^{100} - 1)}{9} - 100\right]$

$= \frac{20}{27}\left[(10^{100} - 1) - 90\right]$

$= \frac{20}{3}[111.....(100 \text{ times}) - 10]$

$= \frac{20}{3}\left[\underbrace{111.....1}_{98 \text{ times}}01\right]$

$= 20[37037037......370367]$ '37' : 32 times

$= [74........074\ 074\ 7340]$ '74' : 32 times

So '7' occur : 33 times.

20. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$.

Ans. (17)

$n^2 - 15n - 27 > 0 \quad \forall\ n \geq 17$

Also $n^2 - 15n - 27 = \left(n - \frac{15}{2}\right)^2 - \frac{333}{4}$

So, if n is a 3 digit number $n^2 - 15n - 27$ can't be product of 3 digits.

So, n is a two digit number.

So, $n^2 - 15n - 27 \leq 81$ [For maximum product of two 1 digit numbers]

$n^2 - 15n - 108 \leq 0$

So, $n \in \{17, 18, 19, 20\}$

For these values

$n^2 - 15n - 27 = 7$ when n = 17

$\neq 8$ n = 18

$\neq 9$ n = 19

$\neq 0$ n = 20

So, n = 17 is only possible value.

21. Let ABC be an acute-angled triangle and let H be its orthocenter. Let G_1, G_2 and G_3 be the centroids of the triangles HBC, HCA and HAB, respectively. If the area of triangle $G_1G_2G_3$ is 7 units. What is the area of triangle ABC?

Ans. (63)

Area (ΔABC)

= 4 Area (ΔDEF)

= 4[Area (ΔHDF)

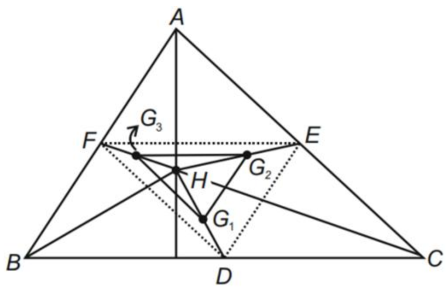
+ Area (ΔHDE)

+ Area (ΔHEF)]

$= 4\left[\frac{9}{4}\{\text{Area } (\Delta HG_1G_3 + \text{Area } (\Delta HG_1G_2) + \text{Area } (\Delta HG_2G_3\}\right]$

$= 9[\text{Area } \Delta G_1G_2G_3] \quad \left[\frac{\text{Area } (\Delta HED)}{\text{Area } (\Delta HG_2G_1)} = \left(\frac{3}{2}\right)^2\right]$

$= 9 \times 7 = 63$



$\frac{HE}{HG_2} = \frac{3}{2}$

22. A positive integer k is said to be good if there exists a partition of $\{1,2,3,\ldots,20\}$ in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good number are there?

Ans. (6)

Sum of all numbers

1 + 2 + 3 + ... + 20 = 210

So we can possibly be divide in

2 groups (sum of each 105)

3 groups (sum of each 70)

5 groups (sum of each 42)

6 groups (sum of each 35)

7 groups (sum of each 30)

10 groups (sum of each 21)

Sum of each can't be less than 20.

Now for 2, 5, 10 groups, divide numbers in 10 pairs :

(1, 20), (2, 19), (3, 18),.............. (10, 11)

Put 5 pairs each for groups of 2

Put 2 pairs each for groups of 5

Put 1 pair each for groups of 10

For 3 groups distribute in following manner.

Subset 1		**Subset 2**	**Subset 3**
20		19	18
15		16	17
14		13	12
9		10	11
8	To S–2	7	6
3		4	5
	To S–1	(2)	(1)

For 6 groups (20, 9, 1, 2, 3), (19, 12, 4), (18, 11, 6),
(17, 10, 8), (16, 14, 5), (15, 13, 7)

For 7 groups (20, 10), (19, 9, 2), (18, 8, 4), (17, 7, 6)
(16, 13, 1), (15, 12, 3), (14, 11, 5)

Total number of good number = 6.

23. What is the largest positive integer n such that $\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \ge n(a+b+c)$ holds for all positive real numbers a, b, c.

Ans. (14)

Using Titu's lemma

$$\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq \frac{(a+b+c)^2}{\left[\frac{b}{29}+\frac{c}{31}+\frac{c}{29}+\frac{a}{31}+\frac{a}{29}+\frac{b}{31}\right]}$$

$$=\frac{(a+b+c)^2}{(a+b+c)\left[\frac{1}{29}+\frac{1}{31}\right]}$$

$$=\frac{29\times 31}{60}(a+b+c)$$

= 14.983 (a + b + c)

So, given inequality holds if $n \leq 14$

Largest value of n = 14.

24. If N is the number of triangle of different shapes (i.e., not similar) whose angles are all integers (in degrees). What is N/100?

Ans. (27)

Method – 1

No. of Δ with smallest angle $1^\circ = 89$

$[(1^\circ, 1^\circ, 179^\circ)\ (1^\circ, 2^\circ, 178^\circ), ... (1^\circ, 89^\circ, 90^\circ)]$

No. of Δ with smallest angle $2^\circ = 88$

$[(2^\circ, 2^\circ, 176^\circ)\ (2^\circ, 3^\circ, 175^\circ), ... (2^\circ, 89^\circ, 89^\circ)]$

No. of Δ with smallest angle $3^\circ = \frac{172}{2} = 86$

No. of Δ with smallest angle $4^\circ = \frac{169-1}{2}+1 = 85$

No. of Δ with smallest angle $5^\circ = \frac{166}{2} = 83$

No. of Δ with smallest angle $60^\circ = 1$

$\therefore$ Total no. of $\Delta = (89+88)+(86+85)+(83+82)+......(2+1)$

$=\frac{30}{2}[(89+88)+92+1)]$

$N = 2700$

$\therefore \frac{N}{200} = 27$

Method – 2

$x+y+z=180$

Total solution = ${}^{179}C_2$ ...... (i)

Out of these, number of cases where exactly two of three angles are equal.

i.e. 2x + z = 180 are 88

each of these case are counted thrice $[x = 1^\circ, 2^\circ, ...59^\circ, 61^\circ, ...89^\circ]$ in eq. (i)

Also, number of cases where x = y = z is 1.

So, Total unique cases

$$=\frac{1}{6}\left[{}^{179}C_2-3\times 88-1\right]+88+1$$

= 2700

When $x\neq y\neq z$. Such cases $(\text{ex. }70^\circ, 80^\circ, 30^\circ)$ are counted 3! Times in (1) whereas we will have only one Δ for every such 3! cases.

25. Let T be the smallest positive integer which, when divided by 11,13, 15 leaves remainders in the sets {7, 8, 9}, {1, 2, 3}, {4, 5, 6} respectively. What is the sum of the squares of the digits of T?

Ans. (81)

Numbers between 1 and 143(LCM(11, 13)) that satisfies the condition of remainders of 11 and 13 are :

29, 40, 41, 42, 53, 106, 107, 118, 119,

None of these number satisfies the condition of remainder of 15.

So adding 143 (LCM(11, 13)) to these numbers to obtain a number which satisfies condition of remainder of 15 also.

The smallest such number : 41 + 143 = 184.

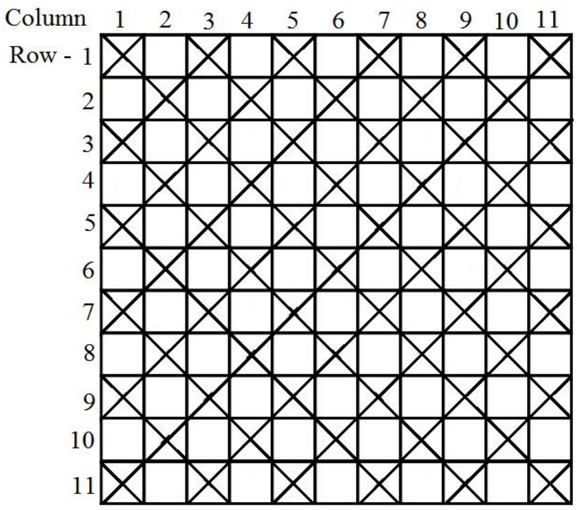
$\therefore T=184$

Sum of squares = 1 + 64 + 16 = 81

26. What is the number of ways in which one can choose 60 unit squares from a 11 × 11 chessboard such that no two chosen squares have a side in common?

Ans. (61)

Maximum of such squares can be chosen when 6 square from row 1, 5 from row 2. 6 from row 3, ... 6 from row 11 are chosen. i.e. maximum of 61 squares can be chosen. If we leave any one of 61 squares in 61 ways we can have 60 required squares.



Also if we chose 6 squares from row 2, row 4, row 6, row 10. Then 5 squares each will have to be chosen from row 1, row 3, ... row 11 to chose 60 squares.

So, total ways = 60 + 1 = 61.

27. What is the number of ways in which one can colour the squares of a 4 × 4 chessboard with colours red and blue such that each row as well as each column has exactly two red squares and two blue squares?

Ans. (90)

(1) Colour 1st row in ${}^4C_2 = 6$ ways

[Say R B R B]

R	B	R	B
B			
B			
R			

(2) Now fill remaining 3 elements of 1st column in 3 ways.

(3) Now, fill that row which has same 1st element as that of 1st row.

R	B	R	B
B	R	B	R
B	R	B	R
R	B	R	B

C – 1 in this both there rows are identical so no. of ways in this case : 1

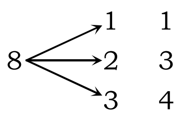
R	B	R	B
B	R	R/B	R
B	R	B/R	R
R	B	B	R

C – 2 in this case exactly two elements of these two rows are matching in 2 ways. The other two rows can be filled in 2 ways.

So, total ways $= 6 \times 3[1 + 2 \times 2] = 90$.

28. Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates,. Find the sum of the digits of N.

Ans. (24)



[Such that each child get at least one chocolate and no two get same number of chocolate]

The possible ways to distribute :

1, 2, 5 or 1, 3, 4 number of chocolates to the children.

Number of ways to form groups.

$$\frac{8!}{1!2!5!} + \frac{8!}{1!3!4!} = 448$$

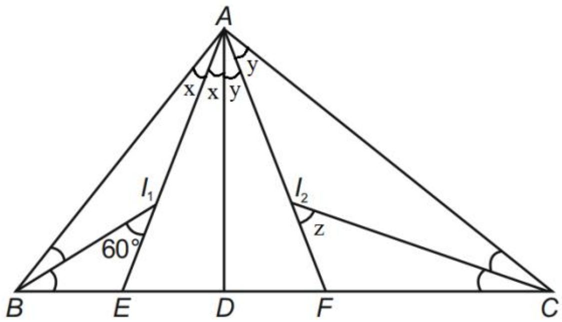
and number of ways to distribute : $448 \times 3! = 2688$

$\therefore$ sum $= 24$

29. Let D be an interior point of the side BC of a triangle ABC. Let I_1 and I_2 be the incentres of triangles ABD and ACD respectively. Let AI_1 and AI_2 meet BC in E and F respectively. If $\angle BI_1E = 60^0$. What is the measure of $\angle CI_2F$ in degrees.

Ans. (30)

From figure



$60^\circ = \frac{B}{2} + x$

$z = \frac{C}{2} + y$

$60 + z = \frac{B}{2} + \frac{C}{2} + (x + y)$

$= \frac{B}{2} + \frac{C}{2} + \frac{A}{2}$

$= 90^\circ$

$\therefore \quad z = 30^\circ$

30. Let $P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ be a polynomial in which a_i is a non-negative integer for each $i \in \{0,1,2,3,...,n\}$. If P(1) = 4 and P(5) = 136, what is the value of P(3)?

Ans. (34)

$a_0 + a_1 + ...a_n = 4$

So, $n \le 4$

n = 4 only if $a_0 = a_1 = a_2 = a_3 = 1$

But then P(5) can't be 136

n = 3,

$P(x) = x^3 + 2x + 1$

Satisfies $P(1) = 4$ and $P(5) = 136$

$\therefore P(3) = 27 + 6 + 1 = 34$